Binomial Expansions

Expansions of $(a + b)^n$

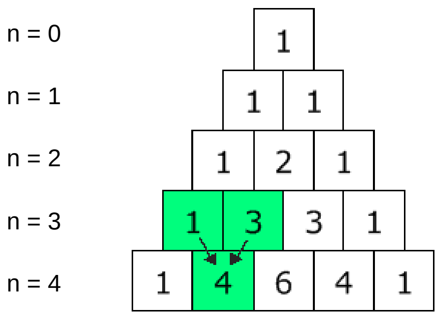
n	$(a + b)^n$	Expansion
n = 0	$(a + b)^0$	1
n = 1	$(a + b)^1$	$1a + 1b$
n = 2	$(a + b)^2$	$1a^2 + 2ab + 1b^2$
n = 3	$(a + b)^3$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$
n = 4	$(a + b)^4$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$
n = 5	$(a + b)^5$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Patterns in the Powers of the expansions

1) The number of terms in the expansion is n + 1.
2) The power of a starts with a^n and decrease by one in each successive term and ends with $a^0 = 1$.
3) The power of b starts with $b^0 = 1$ and increase by one in each successive term and ends with b^n.
4) In each term, the powers of a and b sum to n. (Dimensionally correct).
5) The powers of the first and last terms are always n. i.e. a^n and b^n.

Patterns in the Coefficients of the expansions

6) The coefficients of the terms form a symmetrical pattern.
7) The first & last coefficients are always 1.
8) The second and penultimate coefficients are always n.
9) The remaining coefficients form a less obvious pattern but can be found using Pascal's Triangle for small n, (or the ${}^nC_r$ formula otherwise).



The first and last digits in each row are 1.

The other terms are each the sum of the two terms immediately above them in the triangle.

Example 1: Expand $(2 + x)^3$

Compare with $(a + b)^3$ and let $a = (2)$ and $b = (x)$ where $n = 3$.

Now $(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

So $(2 + x)^3 = 1(2)^3 + 3(2)^2(x) + 3(2)(x)^2 + 1(x)^3$

$= 8 + 3(4)x + 3(2)x^2 + x^3$

$= 8 + 12x + 6x^2 + x^3$

Example 2: Expand $(2 + 5x)^4$

Compare with $(a + b)^4$ and let $a = (2)$ and $b = (5x)$ where $n = 4$.

Now $(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

So $(2 + 5x)^4 = 1(2)^4 + 4(2)^3(5x) + 6(2)^2(5x)^2 + 4(2)(5x)^3 + 1(5x)^4$

$= 16 + 4(8)(5)(x) + 6(4)(5)^2(x)^2 + 4(2)(5)^3(x)^3 + (5)^4(x)^4$

$= 16 + 32(5)(x) + 24(25)(x^2) + 8(125)(x^3) + (625)(x^4)$

$= 16 + 160x + 600x^2 + 1000x^3 + 625x^4$

Rules of Indices
$(p\times q)^m = p^m \times q^m$

Example 3: Expand $(1 – 2x)^4$

Compare with $(a + b)^4$ and let $a = (1)$ and $b = (– 2x)$ where $n = 4$.

Now $(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

So $(1 – 2x)^4 = 1(1)^4 + 4(1)^3(– 2x) + 6(1)^2(– 2x)^2 + 4(1)(– 2x)^3 + 1(– 2x)^4$

$= 1 + 4(– 2)(x) + 6(– 2)^2(x)^2 + 4(– 2)^3(x)^3 + (– 2)^4(x)^4$

$= 1 – 8(x) + 6(4)(x^2) – 4(8)(x^3) + (16)(x^4)$

$= 1 – 8x + 24x^2 – 32x^3 + 16x^4$

Rules of Indices
$(p\times q)^m = p^m \times q^m$

Alternating
+ - signs
with even
& odd
powers

Example 4:

Using the first 4 terms of the expansion $(1 + x)^6$ estimate 0.9^6 to 2 significant figures.

$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + \cdots$

Let x = – 0.1 so that 1 + x = 0.9 then

$(1 - 0.1)^6 = 1 + 6(-0.1) + 15(-0.1)^2 + 20(-0.1)^3 + \cdots$

$(0.9)^6 = 1 - 0.6 + 15(0.01) - 20(0.001) + \cdots$

$(0.9)^6 = 0.4 + 0.15 - 0.020 + \cdots$

$(0.9)^6 = 0.55 - 0.02 + \cdots$

$0.9^6 \approx 0.53$ (2 significant figures)

Alternating + - signs with even & odd powers

{The real answer is $0.9^6 = 0.531441$}

Question 1:

Expand and simplify the following:

a) $(2 + x)^3$

b) $(1 + x)^7$

c) $(3 + 2x)^4$

d) $(1 - x)^9$

e) $(1 - 3x)^5$

Pascal’s Triangle

1											
1	1										
1	2	1									
1	3	3	1								
1	4	6	4	1							
1	5	10	10	5	1						
1	6	15	20	15	6	1					
1	7	21	35	35	21	7	1				
1	8	28	56	70	56	28	8	1			
1	9	36	84	126	126	84	36	9	1		
1	10	45	120	210	252	210	120	45	10	1	
1	11	55	165	330	462	462	330	165	55	11	1

Question 2:

Using the first 4 terms of the expansion $(1 + x)^5$ estimate 1.1^5 to 3 significant figures.

ANSWERS

1a) $(2 + x)^3 = 1(2)^3 + 3(2)^2x + 3(2)x^2 + 1x^3$

$= 8 + 12x + 6x^2 + x^3$

1b) $(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

1c) $(3 + 2x)^4 = 1(3)^4 + 4(3)^3(2x) + 6(3)^2(2x)^2 + 4(3)(2x)^3 + 1(2x)^4$

$= 81 + (4\times27\times2)x + (6\times9\times4)x^2 + (4\times3\times8)x^3 + 16x^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$

1d) $(1 - x)^9 = 1 + 9(-x) + 36(-x)^2 + 84(-x)^3 + 126(-x)^4 + 126(-x)^5 +$

$+ 84(-x)^6 + 36(-x)^7 + 9(-x)^8 + 1(-x)^9$

$= 1 - 9x + 36x^2 - 84x^3 + 126x^4 - 126x^5 + 84x^6 - 36x^7 + 9x^8 - x^9$

1e) $(1 - 3x)^5 = 1 + 5(-3x) + 10(-3x)^2 + 10(-3x)^3 + 5(-3x)^4 + 1(-3x)^5$

$= 1 - (5\times3)x + (10\times9)x^2 - (10\times27)x^3 + (5\times81)x^4 - 243x^5$

$= 1 - 15x + 90x^2 - 270x^3 + 405x^4 - 243x^5$

Question 2

$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + \cdots$

Let $x = 0.1$

$(1 + 0.1)^5 = 1 + 5(0.1) + 10(0.1)^2 + 10(0.1)^3 + \cdots$

$(1 + 0.1)^5 = 1 + 0.5 + 10(0.01) + 10(0.001) + \cdots$

$(1 + 0.1)^5 = 1 + 0.5 + 0.10 + 0.010 + \cdots$

$(1 + 0.1)^5 = 1.5 + 0.11 + \cdots$

so $1.1^5 \approx 1.61$ (3 significant figures)

{The real answer is $1.1^5 = 1.61051$}