Disguised Quadratics

Functions and roots

You need to be able to use function notation and to be able to solve quadratic equations in a **function of the unknown**. Here is an example:

You don't know how to solve a quartic equation like this, but if you write $u = x^2$ then $u^2 = (x^2)^2 = x^4$.

$x^4 - 7x^2 + 12 = 0$

$u^2 - 7u + 12 = 0$ — Substitute $u = x^2$ to form a quadratic equation in u.

$(u - 3)(u - 4) = 0$

$u = 3$ or $u = 4$

Use the substitution to convert back to x. — $x^2 = 3$ $\quad x^2 = 4$

The equation has **four solutions**. — $x = \pm\sqrt{3}$ $\quad x = \pm 2$

You could also write
$(x^2)^2 - 7(x^2) + 12 = 0$
$(x^2 - 3)(x^2 - 4) = 0$

Worked example

Solve the equation $3x + \sqrt{x} - 2 = 0$ **(4 marks)**

Let $u = \sqrt{x}$

$3x + \sqrt{x} - 2 = 0$

$3u^2 + u - 2 = 0$

$(3u - 2)(u + 1) = 0$

$3u - 2 = 0$ or $u + 1 = 0$

$u = \frac{2}{3}$ or $\cancel{u = -1}$

Using $u = \sqrt{x}$

$x = u^2$

$x = \frac{4}{9}$

Problem solved!

$x = (\sqrt{x})^2$ so you could write the equation as:

$3(\sqrt{x})^2 + \sqrt{x} - 2 = 0$

This is a **quadratic equation in** $\sqrt{x}$. The safest way to solve equations like this is to use the substitution $u = \sqrt{x}$.

$\sqrt{x}$ is the positive square root of x, so it can only take positive values. You need to ignore the solution $u = -1$.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Domain

Functions will usually be defined for a given domain. This is the set of values that can be used as the input to the function. The domain of this function is all the positive **real numbers** ($\mathbb{R}$).

$g(x) = 2x^2 - 5x - 3, x \in \mathbb{R}, x > 0$

The **roots** of a function, $g(x)$, are the values of x for which $g(x) = 0$. You might need to consider the domain when finding the roots of a function.

Worked example

$g(x) = 2x^2 - 5x - 3, x \in \mathbb{R}, x > 0$

Show that $g(x)$ has exactly one root and find its value. **(3 marks)**

Set $g(x) = 0$ to find roots:

$2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$\cancel{x = -\frac{1}{2}}$ or $x = 3$

The domain is $x > 0$, so $x = -\frac{1}{2}$ is not a root. The only root is $x = 3$.

Now try this

1 Solve
 (a) $x^4 - 3x^2 - 4 = 0$ **(4 marks)**
 (b) $8x^6 + 7x^3 - 1 = 0$ **(4 marks)**
 (c) $x + 10 = 7\sqrt{x}$ **(4 marks)**

2 Solve the equation $4\sqrt{x} + x = 3$, giving your answer in the form $a - b\sqrt{7}$, where a and b are integers to be found. **(5 marks)**

3 $f(x) = x^4 - 4x^2 - 5, x \in \mathbb{R}, x < 0$

Show that $f(x)$ has only one root and determine its exact value. **(4 marks)**

Look at the domain of the function carefully. $x^4 - 4x^2 - 5 = 0$ has **two real solutions** but only one of them is a root of $f(x)$.

5. Functions and roots

1 (a) $x^4 - 3x^2 - 4 = 0$

$(x^2)^2 - 3x^2 - 4 = 0$

$(x^2 + 1)(x^2 - 4) = 0$

$\cancel{x^2 = -1}$ or $x^2 = 4$ so $x = 2$ or $x = -2$

(b) $8x^6 + 7x^3 - 1 = 0$

$8(x^3)^2 + 7x^3 - 1 = 0$

$(8x^3 - 1)(x^3 + 1) = 0$

$x^3 = -1$ or $x^3 = \frac{1}{8}$ so $x = -1$ or $x = \frac{1}{2}$

(c) $x + 10 = 7\sqrt{x}$

$(\sqrt{x})^2 - 7\sqrt{x} + 10 = 0$

$(\sqrt{x} - 5)(\sqrt{x} - 2) = 0$

$\sqrt{x} = 5$ or $\sqrt{x} = 2$ so $x = 25$ or $x = 4$

2 $4\sqrt{x} + x = 3$

$(\sqrt{x})^2 + 4\sqrt{x} - 3 = 0$

$(\sqrt{x} + 2)^2 - 4 - 3 = 0$

$\sqrt{x} + 2 = \pm\sqrt{7}$

$\sqrt{x} = -2 \pm \sqrt{7}$

so $\sqrt{x} = -2 + \sqrt{7}$ or $\cancel{\sqrt{x} = -2 - \sqrt{7}}$

$x = (-2 + \sqrt{7})^2$

$= 11 - 4\sqrt{7}$

$a = 11, b = 4$

3 $f(x) = 0$

$x^4 - 4x^2 - 5 = 0$

$(x^2 - 5)(x^2 + 1) = 0$

$x^2 = 5$ or $\cancel{x^2 = -1}$

$x \in \mathbb{R}, x < 0$ so the only root is $x = -\sqrt{5}$